Details of the proof in

Vaccinating the oldest against Covid-19 saves both the most lives and most years of life

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The hazard at age x is denoted by $h(x)$ and is assumed to be positive at all ages. Let $H(x) = \int_0^x h(a)da$ denote the cumulative hazard. Because h is positive, H is a strictly increasing function. Life expectancy at age x is given by

$$e(x) = \frac{1}{e^{-H(x)}} \int_x^\infty e^{-H(y)} dy$$

where the denominator $e^{-H(x)}$ gives the proportion surviving to age x and the integral sums up years lived by survivors to ages y .

We assume that the years-of-life-saved is proportional to the product $h(x)e(x)$, and we seek conditions guaranteeing that

$$g(x) = h(x)\, e(x) = h(x)e^{H(x)} \int_x^\infty e^{-H(y)}\, dy \tag{1}$$

is an increasing function.

Proposition: Suppose that the hazard $h(x)$ is twice differentiable for all ages x and that

$$\left(\frac{d \, \log h(x)}{dx}\right)^2 > \frac{d^2 \, \log h(x)}{dx^2}.$$

Then $g(x) = h(x)e(x)$ is monotone increasing in age.

Proof: The proof proceeds in three steps. First, we find a new expression for g as a function of cumulative hazards, rather than as a function of age. We call this new function $\tilde{g}$. Next we use tools from analysis to show that the hypothesis of the theorem implies that $\tilde{g}$ is monotone increasing. Lastly, we show that because $\tilde{g}$ is monotone increasing, g must also be monotone increasing.

Because H is strictly increasing, there is a one-to-one correspondence between each age x and the cumulative hazard H at that age. This means that each value of H can be associated with a unique age x, and H is an invertible function. Let J be the inverse of H, so that $J(w)$ is the age at which cumulative hazards have reached level w. Define new variables u and t by $u = H(x)$ and $t = H(y) - H(x) = H(y) - u$, and observe that $\frac{du}{dx} = h(x)$. Let $\lambda = h \circ J$ be the composition of the hazard h with the inverse function J. Then $\lambda(w)$ tells us the hazard experienced at the age when cumulative hazards are equal to w. In particular, $\lambda(u) = h(x)$ and $\lambda(u + t) = h(y)$.

Now define a new function $\tilde{g}$ by $\tilde{g}(u) = g(x)$. The factor $h(x)e^{H(x)}$ appearing before the integral in equation (1) can be rewritten as $\lambda(u)e^u$. Using $\frac{d}{dy}H(y) = h(y)$, we change the variable of integration in equation (1) from y to t, and then by absorbing $\lambda(u)e^u$ into the integral we find that the value $\tilde{g}(u)$ corresponding to $g(x)$ is

$$\tilde{g}(u) = \int_0^\infty \frac{\lambda(u)\, dt}{\lambda(u + t)\, e^t}. \tag{2}$$

To establish that $\tilde{g}$ is monotone increasing, we show that $\tilde{g}(u + \delta) > \tilde{g}(u)$ for any $\delta > 0$. Observe that

$$\tilde{g}(u+\delta) = \int_0^\infty \frac{\lambda(u + \delta)\, dt}{\lambda(u + \delta + t)\, e^t} = \int_0^\infty \left(\frac{\lambda(u + \delta)\lambda(u + t)}{\lambda(u)\, \lambda(u + t + \delta)}\right) \frac{\lambda(u)\, dt}{\lambda(u + t)\, e^t}. \tag{3}$$

Write Φ for the fraction in brackets inside the last integral of equation (3), and notice that the integral in equation (3) differs from the integral in equation (2) by the factor Φ. We will show that $\Phi > 1$, so that $\tilde{g}(u + \delta) > \tilde{g}(u)$.

Define ages x_1, x_2, x_3, and x_4 by $x_1 = J(u)$, $x_2 = J(u+\delta)$, $x_3 = J(u+t)$, and $x_4 = J(u + t + \delta)$, and put $f(x) = \log h(x)$. From the definition of λ in terms of the inverse function J, we can write $\log \lambda(u) = \log h(J(u)) = \log h(x_1) = f(x_1)$, and similarly $\log \lambda(u + \delta) = f(x_2)$, $\log \lambda(u + t) = f(x_3)$, and $\log \lambda(u + t + \delta) = f(x_4)$. Thus

$$\log \Phi = (f(x_2) - f(x_1)) - (f(x_4) - f(x_3)).$$

For small δ we have

$$\delta = \int_{x_1}^{x_2} h(y)\, dy \;\approx\; (x_2 - x_1)h(x_1)$$

and similarly $\delta \approx (x_4 - x_3)h(x_3)$.

By first order Taylor expansions we approximate $f(x_2)-f(x_1)$ with $(x_2 - x_1)\frac{df}{dx}|_{x_1}$, and similarly $f(x_4) - f(x_3)$ with $(x_4 - x_3)\frac{df}{dx}|_{x_3}$. Letting $F(x) = -\frac{1}{h(x)}\frac{df(x)}{dx}$, we then have

$$\begin{aligned}
\log \Phi &= (f(x_2) - f(x_1)) - (f(x_4) - f(x_3)) \\
&\approx (x_2 - x_1)\frac{df}{dx}|_{x_1} - (x_4 - x_3)\frac{df}{dx}|_{x_3} \\
&= \frac{\delta}{h(x_1)}\frac{df}{dx}|_{x_1} - \frac{\delta}{h_{x-3}}\frac{df}{dx}|_{x_3} \\
&= \delta(F(x_3) - F(x_1)).
\end{aligned}$$

It then follows from the fundamental theorem of calculus that

$$\begin{aligned}
\log \Phi &\approx \delta \int_{x_1}^{x_3} \frac{d}{dx}F(x)\, dx \\
&= \delta \int_{x_1}^{x_3} \frac{-d}{dx}\left(e^{-f(x)}\frac{df}{dx}\right) dx.
\end{aligned}$$

The integrand $\frac{-d}{dx}\left(e^{-f(x)}\frac{df}{dx}\right)$ is equal to $e^{-f}\left((\frac{df}{dx})^2 - \frac{d^2f}{dx^2}\right)$. By assumption, $(\frac{df}{dx})^2 - \frac{d^2f}{dx^2}$ is greater than zero, and thus $\log\Phi$ is positive and $\tilde{g}(u+\delta) > \tilde{g}(u)$. This inequality holds for all u and small $\delta > 0$, and so $\tilde{g}$ is an increasing function.

Lastly, recall that $\frac{du}{dx} = h(x)$ is assumed to be positive. Because $\tilde{g}$ is an increasing function, $\frac{d\tilde{g}}{du} > 0$. Since $\tilde{g}(u) = g(x)$, we have

$$\frac{d}{dx}g(x) = \frac{d}{dx}\tilde{g}(u) = \frac{d\tilde{g}}{du}\frac{du}{dx} = \frac{d\tilde{g}}{du}h(x) > 0,$$

and hence g is an increasing function. Q.E.D.